Fixed-Income Securities, Interest Rates and Term Structure of Interest Rates

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These lecture notes are based on:

- the lecture notes on Portfolio Selection by Prof. Friedmann, University of Saarland
- *"The Econometrics of Financial Markets"*, Campbell, Lo and MacKinlay, 1997, Princeton University Press (Chapter 10)

Outline

Outline

Definition

- **Fixed-income securities** are default-free bonds with payments that are fully specified in advance.
- fixed-income markets $\neq$ equity markets
- Examples for fixed-income securities include Treasury securities - bills, notes, bonds (US), Staatsanleihen - Bundesobligationen/-anleihen (Germany), etc.

Notation

- Price of a security at time t:

$$P_t, \quad P_t > 0$$

- Future payment at time $t + i$:

$$C_{t+i}, \quad i \in I \subseteq \mathbb{N}, \quad C_{t+i} \geq 0$$

- Payments take place at *discrete* times $t \in \mathbb{N}_0$

- Length of a time interval from t to $t + n$ ($n \in \mathbb{N}$): n (basis) periods

Discrete vs. continuous returns: Discrete returns

If not explicitly specified, returns are based on a single holding period. The **discrete return** R of a bond with price $P_t > 0$ at time t and future payments $C_{t+i},\ i \in I \subseteq \mathbb{N}$, $C_{t+i} \geq 0$, $\sum_{i \in I} C_{t+i} > 0$, is defined as the interest rate R, for which the price P_t equals the present value of the future payments:

$$P_t = \sum_{i \in I} \frac{C_{t+i}}{(1+R)^i}, \quad R > -1,$$

Interest payments take place at discrete points of time.
Special case: Discrete return for a single payment at time $t+n$

$$1 + R = \left(\frac{C_{t+n}}{P_t} \right)^{\frac{1}{n}} \quad \text{and} \quad C_{t+n} = P_t\,(1+R)^n.$$

Discrete vs. continuous returns: Continuous returns

How much is the interest rate x_k, which equals the single-period return $1+R$, when the interest payments take place pro rata within the period after $\frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1$ sub-periods?

$$\begin{aligned} (1+\frac{x_k}{k})^k &= 1+R \\ \Leftrightarrow \qquad x_k &= k((1+R)^{\frac{1}{k}}-1), \end{aligned}$$

e.g. $R = 0.21$:

$x_2 = 0.2$, $x_4 = 0.1952354$, $x_{12} = 0.1921424$, $x_{365} = 0.1906701$.

Discrete vs. continuous returns: Continuous returns

Continuous return: r

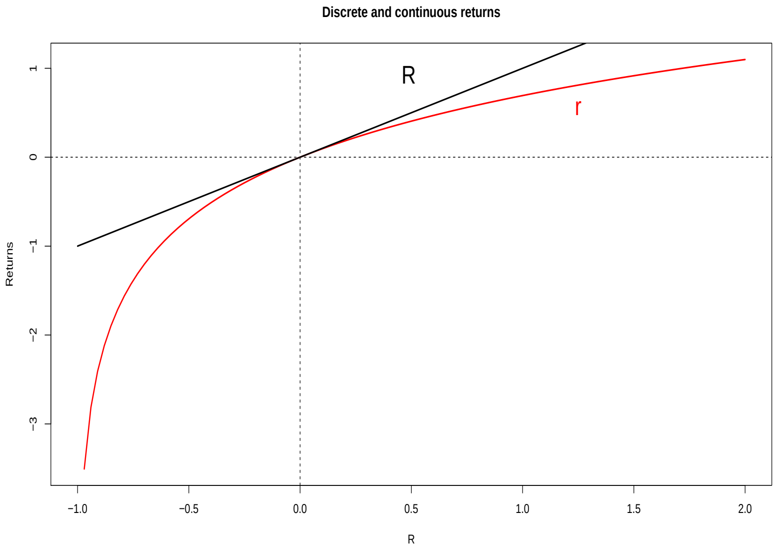
$$\lim_{k\to\infty}(1+\frac{r}{k})^k = 1+R \;\Leftrightarrow\; e^r = 1+R \;\Leftrightarrow\; r = \ln(1+R),$$

e.g. for $R = 0.21 : r = \ln(1.21) = 0.1906204$.

It holds $r = \ln(1+R) \leq R$, Equality is reached when $R = 0$; when $|R| \approx 0$ the difference is minor.

Special case: Continuous return for a single payment at time $t+n$

$$r = \ln(1+R) = \frac{1}{n}\left(\ln C_{t+n} - \ln P_t\right) \quad \text{and} \quad C_{t+n} = P_t\, e^{nr}.$$


Discrete and continuous returns
R
r
Returns
R
1
0
−1
−2
−3
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

Discrete vs. continuous returns: one period returns

Special case:

Discrete and continuous one-period return at time $t+1$ (no additional payment - buy at P_t, sell at $P_{t+1} = C_{t+1}$):

$$\begin{aligned} R_{t+1} &= \frac{P_{t+1}}{P_t} - 1 = \frac{P_{t+1} - P_t}{P_t}. \\ r_{t+1} &= \ln(1 + R_{t+1}) = \ln\left(\frac{P_{t+1}}{P_t}\right) = \ln P_{t+1} - \ln P_t. \end{aligned}$$

Multiple-period returns over k periods

Discrete return $R_t(k)$ and continuous return $r_t(k)$:

$$\begin{aligned} 1 + R_t(k) &= \frac{P_t}{P_{t-k}} \\ r_t(k) &= \ln(1 + R_t(k)) = \ln\left(\frac{P_t}{P_{t-k}}\right) = \ln P_t - \ln P_{t-k} \end{aligned}$$

with

$$\begin{aligned} 1 + R_t(k) &= \prod_{i=0}^{k-1}(1 + R_{t-i}) \\ r_t(k) &= \sum_{i=0}^{k-1} r_{t-i}. \end{aligned}$$

Comparison of returns for different period lengths

Annualisation:
(Example: daily return R_t resp. r_t, 5 trading days per week, 250 trading days p.a.)
Assumption: Daily resp. weekly returns are constant over the period (a week resp. an year)

Discrete returns

Daily return	R_t	annualized with:	$(1+R_t)^{250}-1$
Weekly return	$R_t(5)$	annualized with:	$(1+R_t(5))^{52}-1$

Continuous returns

Daily return	r_t	annualized with:	$250\, r_t$
Weekly return	$r_t(5)$	annualized with:	$52\, r_t(5)$

Outline

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Definition

Zero-coupon bonds, also called discount bonds, make a single payment at a date in the future known as *maturity date*.

The size of the payment is the *face value* of the bond. For convenience in the following we assume that the face value is always 1 monetary unit (say EURO).

The length of time until maturity is the *maturity* of the bond.

An example for a zero-coupon bond are US Treasury bills.

Yield to maturity

Length of time from today (t) until maturity ($t+n$): maturity n

Payment at t (Price): P_{nt}

Payment at $t+n$ (face value): $P_{0,t+n} = 1$ EURO

Yield to maturity: $Y_{nt} = (\frac{1}{P_{nt}})^{\frac{1}{n}} - 1$ with $P_{nt} = \frac{1}{(1+Y_{nt})^n}$

log yield: $y_{nt} = \ln(1 + Y_{nt})$ with $\ln P_{nt} = -ny_{nt}$

Properties

Elasticity of a variable B with respect to a variable A is defined to be the derivative of B with repect to A, times A/B:

$$\frac{dB}{dA} \cdot \frac{A}{B}$$

Equivalently, it is the derivative of $\ln B$ with respect to $\ln A$.

Elasticity of the price with respect to the yield (maturity n)

$$-n = \frac{d \ln P_{nt}}{d \ln(1 + Y_{nt})}, \text{ i.e. } \quad \frac{dP_{nt}}{P_{nt}} = -n \frac{dY_{nt}}{1 + Y_{nt}} \approx -n \cdot dY_{nt}$$

Yield spread and term structure

Yield spread: $S_{nt} = Y_{nt} - Y_{1t}$ resp. $s_{nt} = y_{nt} - y_{1t}$.

Term structure of interest rates at t: Y_{nt} (or y_{nt}) **as a function of** n

Yield curve (plot of the term structure):

"**normal** yield curve": upward sloping,
" **inverse** yield curve": downward sloping.

Discount bonds: holding-period returns

The **holding-period return** on a bond is the return over some holding period *less* than the bond's maturity.

For convenience, the holding-period is set to a single period.

In that way, the discrete return $R_{n,t+1}$ is given through the price change of an $n-$period bond purchased at time t at price P_{nt} and sold at time $t+1$ at price $P_{n-1,t+1}$:

$$1 + R_{n,t+1} = \frac{P_{n-1,t+1}}{P_{nt}} = \frac{(1+Y_{nt})^n}{(1+Y_{n-1,t+1})^{n-1}}$$

resp. for the continuous holding-period return:

$$r_{n,t+1} = \ln P_{n-1,t+1} - \ln P_{nt} = y_{nt} - (n-1)(y_{n-1,t+1} - y_{nt}).$$

Properties

The **excess return** $r_{n,t+1} - y_{1t}$ depends on the spread s_{nt} and the change in the yield over the holding period:

$$r_{n,t+1} - y_{1t} = s_{nt} - (n-1)(y_{n-1,t+1} - y_{nt})$$

It holds: $\quad y_{nt} = \frac{1}{n}\sum_{i=0}^{n-1} r_{n-i,t+i+1}$ (average holding-period return),

due to the compensation of purchasing and selling activities,

$$\sum_{i=0}^{n-1} r_{n-i,t+i+1} = \ln P_{0,t+n} - \ln P_{nt} = \ln 1 + n y_{nt} = n y_{nt}.$$

Forward rate from $t + n$ until $t + n + 1$ with discount bonds

The **forward rate** is an interest rate on a fixed-income investment to be made in the future.
In particular, the forward rate F_{nt} stands for the interest rate negotiated at time t for an investment in a 1-period discount bond, which (the investment) takes place after n periods, i.e. from $t + n$ until $t + n + 1$.
Consider the following strategy:

1) Buy a discount bond (maturity $n + 1$) at price $P_{n+1,t}$
2) Finance the purchase by going short on x discount bonds (maturity n) at price $P_{n,t}$:

$$xP_{n,t} = P_{n+1,t} \Rightarrow x = \frac{P_{n+1,t}}{P_{n,t}}$$

3) *Implied* forward rate F_{nt} with $(1 + F_{nt})x = 1$.

Forward rate

Arbitrage is, loosely speaking, the possibility of a risk-free profit at zero cost.

On an arbitrage-free market, it hods for the forward rate F_{nt}:

$$1 + F_{nt} = \frac{1}{P_{n+1,t}/P_{nt}} = \frac{(1+Y_{n+1,t})^{n+1}}{(1+Y_{nt})^n},$$

resp. for the continuous forward rate f_{nt}:

$$\begin{aligned} f_{nt} = \ln P_{nt} - \ln P_{n+1,t} &= y_{nt} + (n+1)(y_{n+1,t} - y_{nt}) \\ &= y_{n+1,t} + n(y_{n+1,t} - y_{nt}) \end{aligned}$$

as well as $y_{nt} = \frac{1}{n}\sum_{i=0}^{n-1} f_{it}$ (average forward rate).

Remarks

1) $f_{nt} > 0$ (and also $F_{nt} > 0$), if the discount price P_{nt} falls with increasing maturity n.

2) $f_{nt} > y_{nt}$ and $f_{nt} > y_{n+1,t}$ (resp. $F_{nt} > Y_{nt}$ and $F_{nt} > Y_{n+1,t}$), when the term structure is *normal*.

3) F_{nt} (known and fixed at t) is different from $Y_{1,t+n}$ (unknown at t).

Outline

Introduction and Notation

Basic concepts

- Zero-coupon bonds
- Coupon-bonds
- Estimating the Zero-Coupon Term structure

The Expectation Hypothesis and Interest Rate Forecasts

Definition

Coupon bonds make (fixed) coupon payments of a given fraction of face value at equally spaced dates up to and including the maturity date, when the face value is also paid.

Coupon bonds can be though of as **packages of discount bonds**.

Examples include US Treasury notes and bonds.

Formula

Given face value 1 and a (constant) coupon C, the **payments** at $t+i, i = 1, \ldots, n$ are:

$$K_i = C,\ i = 1, \ldots, n-1,\ \text{and } K_n = 1 + C.$$

The **price** at time t is denoted by P_{Cnt} .

$$P_{Cnt} = \sum_{i=1}^{n} K_i P_{it} = \sum_{i=1}^{n} \frac{K_i}{(1+Y_{it})^i}$$

Per-period yield to maturity

The **per-period yield to maturity** (or internal rate of return) on a coupon bond, Y_{Cnt} is defined as:

$$P_{Cnt} = \sum_{i=1}^{n} \frac{K_i}{(1+Y_{Cnt})^i} = C \sum_{i=1}^{n} \frac{1}{(1+Y_{Cnt})^i} + \frac{1}{(1+Y_{Cnt})^n}$$

In general the equation above cannot be inverted to get an analytical solution for Y_{Cnt}.

Exceptions in two special cases (in each case with solution $Y_{Cnt} = C/P_{Cnt}$):

1) $P_{Cnt} = 1$, the bond is sold at face value (at par) $\Rightarrow Y_{Cnt} = C$.
2) $n = \infty \Rightarrow Y_{C\infty t} = C/P_{C\infty t}$.

For coupon bonds payments take place not only at maturity but also at shorter periods. As a result $Y_{Cnt} = C$ (at par) differs generally from Y_{nt}. Generally it holds:

$$\min_{i=1,\ldots,n} Y_{it} \leq Y_{Cnt} \leq \max_{i=1,\ldots,n} Y_{it}.$$

In particular, for $P_{Cnt} > 1$ and large C, Y_{Cnt} diverges from Y_{nt} in direction short-term interest rates, whereas for $P_{Cnt} < 1$ and small C the per-period yield lies nearer to Y_{nt}.

Zero-coupon and coupon term structure

The definition of the coupon term structure requires $P_{Cnt} = 1$.
The coupon term structure assigns (as a function of the maturity) every n the coupon C of the the the at par traded bond:

$$\begin{aligned}
C(n,t) &:= Y_{Cnt|P=1} = C \qquad \text{with} \\
1 &= C(n,t)\sum_{i=1}^{n}\frac{1}{(1+Y_{it})^i} + \frac{1}{(1+Y_{nt})^n}, \quad \text{resp.} \\
C(n,t) &= \frac{1-\frac{1}{(1+Y_{nt})^n}}{\sum_{i=1}^{n}\frac{1}{(1+Y_{it})^i}} \quad , \quad n = 1, 2, \ldots
\end{aligned}$$

Duration (effective maturity) of a coupon-bond

Example: Consider two coupon-bonds with:

- identical maturity $n = 5$,
- identical return per period yield to maturity 10% and
- coupons $C1 = 0.10$ and $C2 = 0.01$.

The respective prices are then $P_{0.10,5,t} = 1$, $P_{0.01,5,t} = 0.6588$.

For which of the two bonds is the percentage price change due to a change in per period yield larger?

Definition

Basic idea: a measure for the length of time that a holder of a *coupon bond* has invested his money.

Macaulay's **Duration** D_{Cnt} of a coupon bond :

$$D_{Cnt} = \sum_{i=1}^{n} w_i \cdot i \ \leq n, \quad \text{mit} \quad w_i = \frac{\frac{K_i}{(1+Y_{Cnt})^i}}{P_{Cnt}}, \quad \sum_{i=1}^{n} w_i = 1.$$

In particular:

$$P_{Cnt} = 1 \ \Rightarrow \ D_{Cnt} = \sum_{i=1}^{n} \left(\frac{1}{1 + C(n,t)} \right)^{i-1} \leq 1 + \frac{1}{C(n,t)}.$$

D_{Cnt} resp. $D^*_{Cnt} = D_{Cnt}/(1 + Y_{Cnt})$ (**modified duration**) serves the function of a measure of the negative of the **elasticity of a coupon bonds's price** with respect to its gross yield:

$$\frac{dP_{Cnt}}{P_{Cnt}} = - \, D_{Cnt} \, \frac{d(1 + Y_{Cnt})}{(1 + Y_{Cnt})} = - \, D^*_{Cnt} \, dY_{Cnt}.$$

Extensions

Modified duration measures the proportional sensitivity of a bond's price to a small absolute change in its yield.
Example: Let $D^*_{Cnt} = 10$. Then the increase in the yield of 1 basis point causes a 10 basis points *drop* in the bond price.

The concept of (modified) duration implies a *linear* relationship between the change in yield and the change in price.
However, the relationship between the log price and the yield is *convex* $\Rightarrow$ improvement through quadratic approximation.

Quadratic approximation

Second-order Taylor series approximation:

$$P(Y+dY) \approx P(Y) + P'(Y)\cdot dY + \frac{1}{2}P''(Y)(dY)^2, \text{ i.e.}$$

$$\frac{dP_{Cnt}}{P_{Cnt}} \approx \underbrace{\frac{dP_{Cnt}}{dY_{Cnt}}\frac{1}{P_{Cnt}}}_{-D^*_{Cnt}} dY_{Cnt} + \frac{1}{2}\underbrace{\frac{d^2P_{Cnt}}{dY^2_{Cnt}}\frac{1}{P_{Cnt}}}_{K_{Cnt}}(dY_{Cnt})^2,$$

with the so-called **convexity** K_{Cnt} of the coupon bond:

$$K_{Cnt} := \frac{d^2P_{Cnt}}{dY^2_{Cnt}}\frac{1}{P_{Cnt}} = \frac{\sum_{i=1}^{n}\frac{i(i+1)}{(1+Y_{Cnt})^{i+2}}K_i}{P_{Cnt}} > 0.$$

Example

$P_{0.04,5,t} = 1$; Price change following an increase of the per period yield to maturity from 4 % to 5 % , i.e. $dY_{Cnt} = 0.01$.

(1) With $Y_{Cnt} = 0.05$ and $C = 0.04$, the **exact** price of the coupon bond is 0.95671 with $dP_{Cnt}/P_{Cnt} = -0.04329$.

(2) **Lin. Approx.**: $D_{Cnt} = 4.62990$, $D^*_{Cnt} = 4.45182$ and
$dP_{Cnt}/P_{Cnt} \approx -D^*_{Cnt} \cdot dY_{Cnt} = -0.04452$.

(3) **Quadr. Approx.**: $K_{Cnt} = 25.0125$, so that
$dP_{Cnt}/P_{Cnt} \approx -D^*_{Cnt} \cdot dY_{Cnt} + 0.5 \cdot K_{Cnt} \cdot (dY_{Cnt})^2 = -0.04327$.

Outline

Problem statement

Basic problem: extract an implied zero-coupon term structure from the coupon term structure.

Justification: From empirical point of view, there is more data on coupon bonds.

In the following the time subscripts t are omitted to economize on notation.

The theoretical relationship between coupon bonds with prices $P_{C_j,j}$, coupons C_j and maturity $j = 1, 2, \ldots, n$ and the discount bond prices $P_1, \ldots, P_n$ resp. the zero-coupon term structure is given as follows:

$$P_{C_j,j} = C_j P_1 + C_j P_2 + \cdots + (1 + C_j) P_j, \ j = 1, 2, \ldots, n.$$

Linear regression approach

Problem: coupon term structure may be more-than-complete.

Idea: consider a stochastic, bond-specific error term u_j in the linear relationship above.

In particular, let J be the number of all the bonds outstanding at a particular date with prices P_{C_j,n_j}, coupons C_j and maturities n_j, $j = 1, 2, ..., J$. Then the relationship between the coupon bond and zero-coupon bond prices can be modeled by the following cross-sectional **linear regression**:

$$P_{C_j n_j} = C_j P_1 + C_j P_2 + \cdots + (1 + C_j) P_{n_j} + u_j, \text{ mit } j = 1, \ldots, J.$$

Details

Classical assumption: u_j iid with $\mathrm{E}(u_j) = 0$ and $\mathrm{Var}(u_j) = \sigma_j^2 \equiv \sigma^2$. Modification: Heteroskedasticity with $\sigma_j = \sigma s_j$, where s_j can be the bid-ask spread or duration.

Let $N = \max_j n_j$, then $\beta = (P_1, P_2, \ldots, P_N)'$ is the vector with the coefficients. The rang condition for the $(J \times N)$-regression matrix X implies $J \geq N$.

Problem : Too many, unrestricted parameters $P_1, \ldots, P_N$.

Approach 1

Discount bond prices lie on a **polynomial**, e.g.

$$P_n = P(n) = 1 + \theta_1 n + \theta_2 n^2 + \theta_3 n^3, \text{ from which follows}$$

$$P^*_{C_j n_j} = X_{n_j 1}\theta_1 + X_{n_j 2}\theta_2 + X_{n_j 3}\theta_3 + u_j, \quad j = 1, \ldots, J, \text{ with}$$

$$\begin{aligned} P^*_{C_j n_j} &= P_{C_j n_j} - 1 - n_j C_j \\ X_{n_j k} &= n_j^k + C_j \sum_{i=1}^{n_j} i^k, \; k = 1, 2, 3 \end{aligned}$$

With $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ one can calculate $\hat{P}_n$ and the estimated zero-coupon bond term structure.

Approach 2

Discount bond prices lie on a **spline function**.

An rth-order spline is a piecewise rth-order polynomial with $r-1$ continuous derivatives; its rth derivative is a step function. The points where the rth derivative changes discontinuously are known as *knot points*. For $K-1$ subintervals there are K knot points: $K-2$ junctions and 2 endpoints. The spline has $K-2+r$ free parameters, r for the first subinterval and 1 (that determines the unrestricted rth derivative) for each of the following $K-2$ subintervals.

Approach 3

Functions with a better fit for larger n.

Nelson/Siegel-approach for a continuous yield $y(n)$, where $P(n) = e^{-ny(n)}$:

$$y^{\mathrm{NS}}(n) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\alpha_1 n}}{\alpha_1 n} \right) + \beta_2 \left(\frac{1 - e^{-\alpha_1 n}}{\alpha_1 n} - e^{-\alpha_1 n} \right).$$

Parameter interpretation:

$$\begin{aligned} \beta_0 &= \lim_{n\to\infty} y(n) \quad \text{(long-term yield)}, \\ \beta_0 + \beta_1 &= \lim_{n\to 0} y(n) \quad \text{(short-term yield)}. \end{aligned}$$

Svensson's extention:

$$y(n) = y^{\text{NS}}(n) + \beta_3 \left(\frac{1 - e^{-\alpha_2 n}}{\alpha_2 n} - e^{-\alpha_2 n} \right).$$

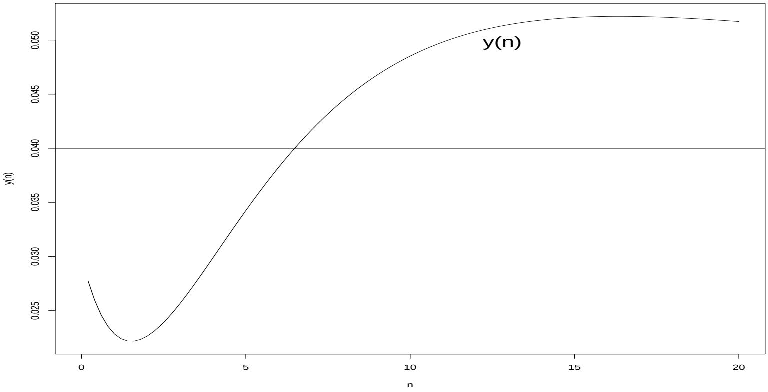
The new term converges for $n \to \infty$ and for $n \to 0$ to zero.
Estimation of $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\alpha}_1, \hat{\alpha}_2$:

$$\min \sum_{j=1}^{J} \frac{(P_{C_j n_j} - \hat{P}_{C_j n_j})^2}{s_j^2} \quad \text{with}$$

$$\hat{P}_{C_j n_j} = C_j \hat{P}(1) + C_j \hat{P}(2) + \cdots + (1 + C_j)\hat{P}(n_j), \quad \hat{P}(n) = e^{-n\,\hat{y}(n)}.$$

Application: Deutsche Bundesbank

Term structure according to Svensson's approach:



$$\beta_0 = 0.04, \beta_1 = -0.01, \beta_2 = -0.1, \beta_3 = 0.1, \alpha_1 = 0.5, \alpha_2 = 0.2$$

Outline

The Expectation Hypothesis

The **term structure** depends on the **expectations of the investors about the future yields** in such a way that (**Hypothesis I**):

$$ny_{nt} \stackrel{!}{=} \mathrm{E}_t\left(y_{1,t} + y_{1,t+1} + \ldots + y_{1,t+n-1}\right).$$

$\mathrm{E}_t\left(\cdot\right)$ denotes the mean conditional on the information $\mathcal{I}_t$ available at time t. (Assumption: the market expectations coincide with $\mathrm{E}_t\left(\cdot\right)$ (rational expectations).)

Interpretation and problems

Interpretation: *The fixed yield on a long-term bond equals the expected yield of consecutive short-term bonds*, i.e. the expected change in the **short-term** interest rates determines the term structure.
Note that the hypothesis is defined for **continuous** returns. It holds:

$$\begin{aligned}(1+Y_{nt})^n &= e^{ny_{nt}} = e^{\mathrm{E}_t\left(\sum_{i=0}^{n-1} y_{1,t+i}\right)} \\ &< \mathrm{E}_t\left(e^{\sum_{i=0}^{n-1} y_{1,t+i}}\right) = \mathrm{E}_t\left(\prod_{i=0}^{n-1}(1+Y_{1,t+i})\right).\end{aligned}$$

This result is due to **Jensen's Inequality**, according to which:

$$E(g(X)) \geq g(E(X)),$$

where X is a random variable and g is a *convex* function.

An alternative specification

In terms of continuous returns, the expectation hypothesis above is equivalent to the following one:
The fixed yield on a short-term bond equals the expected one-period holding-period return on a long-term bond, i.e. the expected change in the **long-term** bonds determines the term structure. Formally (**Hypothesis II**):

$$\begin{aligned} y_{1t} &= \mathrm{E}_t\left(r_{n,t+1}\right) = \mathrm{E}_t\left(\ln P_{n-1,t+1} - \ln P_{n,t}\right) \\ &= \mathrm{E}_t\left(n y_{nt} - (n-1) y_{n-1,t+1}\right). \end{aligned}$$

Turning to discrete returns and applying *Jensen's Inequality*, one gets:

$$1 + Y_{1t} = e^{y_{1t}} = e^{\mathrm{E}_t\left(r_{n,t+1}\right)} < \mathrm{E}_t\left(e^{r_{n,t+1}}\right) = \mathrm{E}_t\left(1 + R_{n,t+1}\right).$$

Equivalence of both hypotheses in the continuous case

For the proof of the equivalence of both hypotheses one uses the rule $\mathrm{E}_t\left(\mathrm{E}_{t+1}\left(\cdot\right)\right) = \mathrm{E}_t\left(\cdot\right)$, which is a special case of the *law of total expectation* $\mathrm{E}(Y) = \mathrm{E}_X(\mathrm{E}(Y|X))$.

From **Hypothesis I** to **Hypothesis II**:

From

$$\begin{aligned} ny_{nt} &= \mathrm{E}_t\left(y_{1,t} + y_{1,t+1} + \ldots + y_{1,t+n-1}\right) \text{ and} \\ (n-1)y_{n-1,t+1} &= \mathrm{E}_{t+1}\left(y_{1,t+1} + \ldots + y_{1,t+n-1}\right) \text{ resp.} \\ \mathrm{E}_t\left((n-1)y_{n-1,t+1}\right) &= \mathrm{E}_t\left(y_{1,t+1} + \ldots + y_{1,t+n-1}\right), \end{aligned}$$

Subtracting the third row from the first one yields:

$$y_{1t} = \mathrm{E}_t\left(ny_{nt} - (n-1)y_{n-1,t+1}\right) = \mathrm{E}_t\left(r_{n,t+1}\right).$$

An alternative proof:

From **Hypothesis II** to **Hypothesis I**:

An iterative application of

$$\begin{aligned} y_{1t} &= \mathrm{E}_t\left(ny_{nt} - (n-1)y_{n-1,t+1}\right) \text{ resp.} \\ ny_{nt} &= y_{1t} + \mathrm{E}_t\left((n-1)y_{n-1,t+1}\right): \end{aligned}$$

delivers the following result:

$$\begin{aligned} ny_{nt} &= y_{1t} + \mathrm{E}_t\left((n-1)y_{n-1,t+1}\right) \\ &= y_{1t} + \mathrm{E}_t\left(y_{1,t+1} + \mathrm{E}_{t+1}\left((n-2)y_{n-2,t+2}\right)\right) \\ &= \ldots \\ &= y_{1t} + \mathrm{E}_t\left(y_{1,t+1} + \ldots + y_{1,t+n-1}\right). \end{aligned}$$

The yield spread s_{nt} as a forecasting tool

The relevance of the yield spread for the forecast of the (short-term) change in the **long-term** interest rates (large n) follows from **Hypothesis II**.
In particular:

$$\begin{aligned} y_{1t} - y_{nt} &= (n-1)\mathrm{E}_t\left(y_{nt} - y_{n-1,t+1}\right), \text{ resp.} \\ \mathrm{E}_t\left(y_{n-1,t+1} - y_{nt}\right) &= \frac{y_{nt} - y_{1t}}{n-1} = \frac{s_{nt}}{n-1}, \end{aligned}$$

or

$$y_{n-1,t+1} - y_{nt} = \frac{s_{nt}}{n-1} + \varepsilon_{nt},$$

with $\mathrm{E}_t\left(\varepsilon_{nt}\right) = 0$.

Relationship with the forward rate

From the original expression

$$ny_{nt} = \mathrm{E}_t\left(y_{1,t} + y_{1,t+1} + \ldots + y_{1,t+n-1}\right),$$

it follows (with the acceptance of this expectation hypothesis) that

$$f_{nt} = (n+1)y_{n+1,t} - ny_{nt} = \mathrm{E}_t\left(y_{1,t+n}\right).$$

In that way the forward rate f_{nt} is a (longterm) forecast for the **short-term** interest rate $y_{1,t+n}$ (and also a forecast for $f_{n-1,t+1}$, since $f_{nt} = \mathrm{E}_t\left(\mathrm{E}_{t+1}\left(y_{1,t+n}\right)\right) = \mathrm{E}_t\left(f_{n-1,t+1}\right)$).

Models and tests with time constant term premia

Forecast for the future **long** rates:

$$\begin{aligned} \mathrm{E}_t\left(y_{n-1,t+1} - y_{nt}\right) &= \alpha_n + \frac{s_{nt}}{n-1} \qquad \text{resp.} \\ y_{n-1,t+1} - y_{nt} &= \alpha_n + \beta_n \frac{s_{nt}}{n-1} + \varepsilon_{nt},\ t = 1, \ldots, T. \end{aligned}$$

Testing the hypothesis $\beta_n = 1$; empirical evidence for $\beta_n < 1$, quite often $\beta_n < 0$, see Campell, Lo, MacKinley (1997), The Econometrics of Financial Markets, Princeton Unversity Press, S. 420-21.)

Forecast for the future **short** rates:

$$\begin{aligned} \mathrm{E}_t\left(y_{1,t+n} - y_{1t}\right) &= \delta_n + (f_{nt} - y_{1t}) \qquad \text{resp.} \\ y_{1,t+n} - y_{1t} &= \delta_n + \beta_n (f_{nt} - y_{1t}) + \varepsilon_{nt},\ t = 1, \ldots, T. \end{aligned}$$

Testing the hypothesis $\beta_n = 1$.
(See Eugene F. Fama (1976): Forward rates as predictors of future spot rates, *Journal of Financial Economics*, Vol. 3, 361-377.)

Note

When the expectation hypothesis are defined for the discrete instead of the continuous returns, **Hypothesis I** and **Hypothesis II** are contradictory.

On the one side:

$$
\begin{aligned}
(1+Y_{nt})^n &\overset{!}{=} \mathrm{E}_t\left((1+Y_{1t})(1+Y_{1,t+1})\cdots(1+Y_{1,t+n-1})\right) \\
&= (1+Y_{1t})\,\mathrm{E}_t\left(\mathrm{E}_{t+1}\left((1+Y_{1,t+1})\cdots(1+Y_{1,t+n-1})\right)\right) \\
&= (1+Y_{1t})\,\mathrm{E}_t\left((1+Y_{n-1,t+1})^{n-1}\right),
\end{aligned}
$$

Note

On the other side:

$$
\begin{aligned}
(1 + Y_{1t}) \;&\overset{!}{=}\; \mathrm{E}_t\left(1 + R_{n,t+1}\right) \;=\; \mathrm{E}_t\left(\frac{P_{n-1,t+1}}{P_{nt}}\right) \\
&=\; (1 + Y_{nt})^n \, \mathrm{E}_t\left(\frac{1}{(1 + Y_{n-1,t+1})^{n-1}}\right),
\end{aligned}
$$

However :

$$
\mathrm{E}_t\left(\frac{1}{(1 + Y_{n-1,t+1})^{n-1}}\right) > \frac{1}{\mathrm{E}_t\left((1 + Y_{n-1,t+1})^{n-1}\right)}.
$$

(Jensen's Inequality)