University of Saarland
Statistics and Econometrics Department
Dipl. Kaufm. Valentin Popov

Exercises on the lecture "Fixed-income securities", Sofia 2012

Exercise 1

Consider the following closing prices of the BMW asset for 4 consecutive trading days: 22,42 EUR on the 24.03.2009; 22,34 EUR on the 25.03.2009; 22,46 EUR on the 26.03.2009 and 23,26 EUR on the 27.03.2009. Calculate the discrete and the continuous returns for the three days 25, 26 and 27.03.2009.

Exercise 2

The investor N.N. plans to invest his capital for 2 years. For this time interval he receives offers from different banks, which differ in their nominal annualized interest rate through different conditions on the interest payment.

i) Bank A: 3,20% interest rate p.a. with monthly interest payment

ii) Bank B: 3,22% interest rate p.a. with quarterly interest payment

iii) Bank C: 3,25% interest rate p.a. with interest payment p.a.

iv) Bank D: 106,60% payment on the invested capital after 2 years

Knowing that the effective interest rate p.a., when the interest payments take place in less than an year, is greater compared to the respective nominal interest rate, N.N. looks for a better comparison of the four investment strategies.

a) Calculate the discrete returns p.a. of the 4 investment strategies.

b) Calculate the two-year discrete returns of the 4 investment strategies.

c) Calculate the continuous returns of the 4 investment strategies

d) Which strategy would you recommend?

Exercise 3

Consider 4 discount bonds A, B, C and D with maturities $n_A = 1$, $n_B = 3$, $n_C = 5$ and $n_D = 7$. The value $P_{0,t+n}$ at maturity (i.e. the face value) for all bonds is 1 € .

a) Calculate the prices of bonds A, B, C and D at emission time t given that all bonds have a 5% p.a. yield.

b) How much do bonds A, B, C and D cost at emission time t when the yield is 4% p.a. resp. 6% p.a.?

c) For which of the bonds is the price at emission time t more susceptible to possible changes in the yield? How can this be explained?

d) How much is the discrete return p.a. of bonds A, B, C and D, when A is issued at price $P_{1,t} = 0,9803922$ €, B is issued at price $P_{3,t} = 0,8638376$ €, C is issued at price $P_{5,t} = 0,7129862$ € and D is issued at price $P_{7,t} = 0,5834904$ €?

e) Draft the term structure of the yields resulting from d). What kind of term structure is this?

Exercise 4

Investor N.N. plans to make a highly profitable investment of 10.000 € for the next two years. His bank makes three offers ($S1$, $S2$ and $S3$) based on this investment:

A1) A $C_2^{S1} = 12.100$ € payment after 2 years,

A2) A $C_1^{S2} = 3.000$ € payment after the first and a $C_2^{S2} = 8.800$ € payment after the second year,

A3) A $C_1^{S3} = 10.000$ € payment after the first and a $C_2^{S3} = 981$ € payment after the second year.

a) Calculate the effective (discrete) return p.a. resulting from the three strategies.

b) Which investment would be preferred if only the sum of the payments is maximized?

c) Following strategies $S2$ and $S3$, capital C_1^{S2} resp. C_1^{A3} stands free after the first year. Investor N.N plans to reinvest this capital (for example in single-period discount bonds). However, the yield $Y_{1,1}$ on this reinvestment is not known at time $t = 0$.
Which of the three strategies is most profitable depending on $Y_{1,1}$?

Exercise 5

On a certain financial market a one-year discount bond is sold at a 5% p.a. yield, while a two-year discount bond is sold at a 6% yield. Moreover, a 9 % p.a. forward rate for a one-year investment after one year is also offered on this market.

a) How much is the forward rate $F_{1,0}$ *implied* by the discount bonds? Is there an opportunity for an arbitrage in this situation?

b) Give — if possible — an arbitrage strategy, for which a profit can be made at the current period.

c) Explain — if possible — which investment in what amount and in which direction (short or long) should be traded so that one can make a profit of 1.000 € at the current period.

Exercise 6

Consider the following discount bonds at time $t = 0$: discount bond A with maturity $n_A = 1$ year and yield $Y_{1,0} = 0,03$, as well as discount bond B with maturity $n_B = 2$ years and yield $Y_{2,0} = 0,1$. Let two coupon bonds E und F (maturity: 2 years, face value: 1 €) be issued at time $t = 0$.

a) On an arbitrage-free market, how much should be the price $P_{0,5;2;0}$ of the coupon bond E, given that its coupon is $C = 0,5$ €?
   How much is the per-period yield $Y_{0,5;2;0}$ in this case?

b) On an arbitrage-free market, how much should be the coupon C of the coupon bond F, given its price $P_{C;2;0} = 0,862393$ €?
   How much is the per-period yield $Y_{C;2;0}$ in this case?

c) Suggest an interpretation of coupon bonds E and F with the help of discount bonds A and B.

Exercise 7

a) Prove the following equality given an arbitrage-free market:

$$(1 + Y_{nt})^n = \prod_{i=0}^{n-1} (1 + F_{it})$$

Provide an economic interpretation of this equation.

b) Prove the following equality given an arbitrage-free market:

$$(1 + Y_{nt})^n = \prod_{i=0}^{n-1} (1 + R_{n-i,t+i+1})$$

Provide an economic interpretation of this equation.

Exercise 8

Use the relationships between Y_{nt}, F_{nt}, $S_{nt}$and P_{Cnt}given in the lecture (under the assumption of an arbitrage-free market) to fill in the missing values in the following table:

n	1	2	3	4
Y_{nt}		5%		
$F_{n-1;t}$		6%		
S_{nt}				3%
$P_{0,1;n;t}$			$1,110438$	

Exercise 9

Generally there is no analytical solution for the per period yield of a coupon bond with maturity $n > 2$ of equation (1) below (see also the lecture). Instead, a numerical solutions are sought. A suitable method is the Newton method, which determines the roots of a differentiable function $f : \mathbb{R} \to \mathbb{R}$. Given a start value x_0, approximative solutions are calculated iteratively from:

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}, \qquad i \in \mathbb{N},$$

A suitable choice of the starting value x_0 often leads to a small number of iterations until a solution at a satisfactory level of approximation is reached. In the following, the Newton method should be applied to determine the per period yield to maturity Y_{Cnt} of a coupon bond with face value 1, maturity $n = 5$, coupon $C = 0,1$ and price $P_{0,1;5;t} = 1,145448780$.

a) Insert the values of C, n and P_{Cnt} in the following equation for Y_{Cnt}:

$$P_{Cnt} = C \sum_{i=1}^{n} \frac{1}{(1 + Y_{Cnt})^i} + \frac{1}{(1 + Y_{Cnt})^n} \tag{1}$$

and find a function $f : \mathbb{R} \to \mathbb{R}$, so that for $x = 1 + Y_{Cnt}$ it holds:

$$f(x) = 0 \Leftrightarrow \text{Equation (1) holds}$$

b) Calculate the first order derivative $f'(x)$.

c) Calculate the start value of the Newton method $x_0 = 1 + \frac{C}{P_{Cnt}}$ (This is the solution for the equation above in some special cases. Hopefully, the root lies near it so that it represents a suitable starting value). Run through three iterations of the Newton method, i.e. calculate x_1, x_2 and x_3.

d) Calculate the approximative value of $P_{0,1;5;t}$ using $\widetilde{Y}_{Cnt} = x_3 - 1$. How would you assess the quality of the approximation?

Exercise 10

Prove the following equations:

(a) $Y_{Cnt|P=1} = C$,

(b) $Y_{C\infty t} = \dfrac{C}{P_{C\infty t}}$.

Exercise 11

The aim of this exercise is to determine the zero-coupon bond term structure Y_{it} $(i = 1, \ldots, n)$ from a given coupon bond term structure $Y_{Cit|P=1}$ $(i = 1, \ldots, n)$.

(a) Prove the following equation: $(1 + Y_{nt})^n = \dfrac{1 + Y_{Cnt|P=1}}{1 - Y_{Cnt|P=1} \sum_{i=1}^{n-1} \frac{1}{(1+Y_{it})^i}}$.

(b) Explain why $Y_{C1t|P=1} = Y_{1t}$ holds.

(c) Using(a) and (b), the zero-coupon bond term structure can be determined iteratively. Given $n = 4$, calculate the zero-coupon bond term structure from the following coupon bond term structure $Y_{C1t|P=1} = 4\%$, $Y_{C2t|P=1} = 5\%$, $Y_{C3t|P=1} = 5,005\%$ and $Y_{C4t|P=1} = 5,01\%$.

(d) Are the zero-coupon resp. coupon bond term structure normal?

Exercise 12

Use the well-known (given arbitrage-free markets)relationships between n (maturity), P_{nt} (price of a discount bond) , P_{Cnt} (price of a coupon bond with coupon C) , Y_{Cnt} (per period yield of the respective coupon bond) and D_{Cnt} (duration of the respective coupon bond) to fill in the table:

n	1	2	3
P_{nt}		0.90	
$P_{0.2;n;t}$			1.38
$Y_{0.2;n;t}$	0.041667		[redacted]
$D_{0.2;n;t}$			[redacted]

Exercise 13

Consider the following yields $Y_{1t} = 2\%$, $Y_{2t} = 2,5\%$ and $Y_{3t} = 3,5\%$. Further consider an *al pari* traded coupon bond with maturity $n = 3$.

(a) Calculate the per period yield of the coupon bond.

(b) Calculate the duration, the modified duration and the convexity of the coupon bond.

(c) Using the results in b), give two approximations of the percentage change in the coupon bond price when the change in the per period yield is 1 %.

Exercise 14

Under the assumption that the expectation hypothesis for the continuous returns holds, prove the following equalities:

$$\mathrm{E}_t\left(y_{1,t+n}\right) = f_{nt} \text{ and } \mathrm{E}_t\left(f_{n-1,t+1}\right) = f_{nt} \ .$$